Quantum Decoherence: A Discussion of Theory and Applications

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***Abstract*—In this report, we investigate the theory and applications of quantum decoherence. We explore the additions decoherence makes to the standard quantum theory and discuss the problems it hopes to address, namely, the problem of definite outcomes and the problem of a preferred basis. From these proposed solutions, we discuss how the decoherence theory can positively and negatively affect interpretations of quantum mechanics. Finally, we examine the effect of decoherence on the reliability of quantum gates used in quantum computing circuits.**

I. Introduction

The study of quantum mechanics focuses largely on entanglement and correlations between different components of a system. Until relatively recently, quantum systems under study were usually analyzed as closed systems (much like how classical systems are often analyzed), with negligible outside interference. However, recent developments - especially the growth of quantum computing - have highlighted the inadequacy of this "closed system" assumption. Modern quantum computers struggle to maintain qubit state coherence for time scales on the order of 10 μs [1]. "Open system" representations are proving to be much more realistic models of quantum systems, with the environment around an observed system being taken heavily into consideration.

Decoherence is the theory that describes the results of this environmental inclusion; the "quantum coherence" of a system "'leaks out' into the environment" [2]. The environment is modeled to have an immense number of degrees of freedom, rendering any information transferred to it as effectively "lost". Therefore, interaction with the environment results in the reduction of information directly accessible by measuring the system. Mathematically, decoherence theory models a quantum system $\mathcal{S}$ as being correlated with an apparatus $\mathcal{A}$ (originally in the "ready" state $|a_r\rangle$) and the environment $\mathcal{E}$ (originally in the state $|e_0\rangle$):

$$\left(\sum_n c_n |s_n\rangle\right)|a_r\rangle|e_0\rangle \to \left(\sum_n |s_n\rangle|a_n\rangle\right)|e_0\rangle \to \sum_n c_n|s_n\rangle|a_n\rangle|e_n\rangle \tag{1}$$

Physically, decoherence can be represented as in Figure 1, with environmental particles like photons "becoming cohered" with the system. The time scale of this interaction is extremely small. For example, using Equation 10 in [2], Zurek calculates that a small macroscopic system (mass of 1 gram and separation of 1 cm) at laboratory conditions would have a decoherence time scale on the order of 10^{40} times smaller than the relaxation time, a characteristic time of the system. Therefore, "even if the relaxation time was of the order of the age of the universe, $\tau_R \sim 10^{17}$ sec, quantum coherence would be destroyed in $\tau_D \sim 10^{-23}$ sec" [2]. This shows that for macroscopic systems, environment-induced decoherence effects are extremely fast; this speed, along with the factors we discuss in Section II.A below, lead to the emergence of classical behavior.

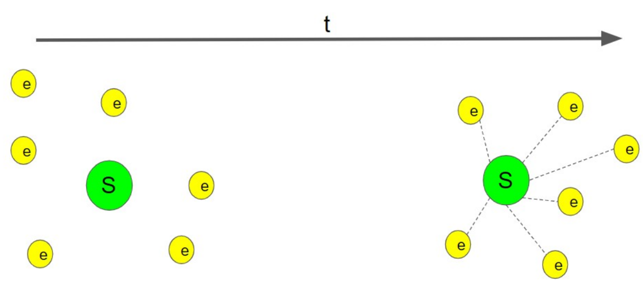

Fig. 1. Decoherence can be represented by interaction of environmental particles e with a quantum system S

In quantum computing, environmental interaction is modeled as a unitary operation U between the "relevant" system density operator ρ and the generic environment density operator ρ_E, as shown in Figure 2. Decoherence, or the loss of information from the relevant system, occurs because the large number of degrees of freedom of the environment render the output environment operator $\rho_{E'}$ effectively unknown.

Fig. 2. In quantum computing, loss of information due to environment-induced decoherence is represented by a unitary operator U

Now, we have multiple perspectives on decoherence and its effects. Using these, we will investigate how it can benefit quantum mechanics as a whole, influence diverse interpretations of quantum mechanics, and describe how quantum circuits perform in noisy environments.

II. Benefits of the Decoherence Program

To quantum computing researchers, decoherence likely holds a negative connotation as environmental noise poses a challenge for robust physical realization of quantum circuits. However, the theory provides strong solutions to problems within the standard quantum mechanical formalism - here we discuss the problem of definite outcomes and the problem of the preferred basis, and the solutions to these problems proposed by decoherence.

A. The Problem of Definite Outcomes

A challenging question any interpretation of quantum mechanics must address is the problem of definite outcomes. We know that a quantum system can be represented by a superposition of states, but when we measure classical observables, we see classical, or "definite", outcomes and not superpositions of multiple possible results. However, we also know that the quantum system is *not* represented by a classical ensemble where we simply do not know the underlying state until it is revealed through measurement. Experiments where we observe the effects of a superposition of states, such as the double slit experiment with electrons, disprove the possibility of such a classical ensemble describing the underlying quantum state [3]. Why then, can we observe only definite outcomes of quantum systems, and not superpositions of multiple results?

Decoherence addresses this issue by including interactions with the environment into analysis of quantum systems. For a generic quantum state $|\psi\rangle$, we can describe the system as a density matrix ρ of the form $\rho = |\psi\rangle\langle\psi|$. In general, ρ can have off-diagonal "interference" elements that represent possible non-classical outcomes. When environment-induced decoherence is introduced to the system, these interference terms quickly dissipate, diagonalizing the density matrix. The remaining diagonal terms can be interpreted as classical probabilities of definite outcomes, answering the question posed above.

To see this more clearly, we present the density-matrix formalism given by Schlosshauer [3]. Without taking environment-induced decoherence into account, the density matrix describing a state $\mathcal{S}$ interacting with an apparatus $\mathcal{A}$ is given by:

$$\hat{\rho}_{\mathcal{SA}} = \sum_{mn} c_m c_n^* \left|s_m\right\rangle \left|a_m\right\rangle \left\langle s_n\right| \left\langle a_n\right| \tag{2}$$

where c_i are constants, $|s_i\rangle$ are bases of the quantum system under study and $|a_i\rangle$ are bases of the apparatus. Note that this matrix representation contains off-diagonal quantum interference terms where $m \neq n$. Terms along this diagonal ($m = n$) represent the classical probabilities of definite outcomes of the system.

When we consider the decoherence effects of the environment, the density matrix becomes:

$$\hat{\rho}_{\mathcal{SAE}} = \sum_{mn} c_m c_n^* \left|s_m\right\rangle \left|a_m\right\rangle \left|e_m\right\rangle \left\langle s_n\right| \left\langle a_n\right| \left\langle e_n\right| \tag{3}$$

where the basis states of the environment $|e_i\rangle$ are added. Since the environment has so many more degrees of freedom than the system, we consider the "local (or reduced)" [3] density matrix that is found by taking the partial trace of $\hat{\rho}_{\mathcal{SAE}}$ with respect to the environment, effectively ignoring all irrelevant degrees of freedom:

$$\hat{\rho}_{\mathcal{SA}} = \mathrm{Tr}_{\mathcal{E}}\left(\hat{\rho}_{\mathcal{SAE}}\right) = \sum_{mn} c_m c_n^* \left|s_m\right\rangle \left|a_m\right\rangle \left\langle s_n\right| \left\langle a_n\right| \left\langle e_n|e_m\right\rangle \tag{4}$$

While the states of environment $|e_i\rangle$ are not necessarily orthogonal by definition, several exact physical models have shown that, due to the many degrees of freedom of the environment, decoherence effects cause the basis states to become approximately orthogonal after a very short time [3]. Therefore, because $\langle e_n|e_m\rangle \approx \delta_{nm}$, the density operator after the system undergoes decoherence is given by:

$$\hat{\rho}_{\mathcal{SA}} \xrightarrow{t} \hat{\rho}_{\mathcal{SA}}^d \approx \sum_{n} |c_m|^2 \left|s_m\right\rangle \left|a_m\right\rangle \left\langle s_n\right| \left\langle a_n\right| \tag{5}$$

The approximate orthogonality of the environment states causes one of the summations to drop out, leaving only the terms where $m = n$. This diagonalized density operator contains only the classical probabilities of definite outcomes, and does not have any off-diagonal interference terms. Thus, only definite outcomes can be observed from this system; *which* definite outcome is observed depends on what quantum mechanical interpretation you use for the collapse of the wavefunction.

B. The Problem of a Preferred Basis

Another problem with the general formalism of quantum mechanics is known as the problem of the preferred basis. The problem exists generally as follows:

If we model a given quantum system $\mathcal{S}$ (that we would like to measure) as being in a composite state $\mathcal{SA}$ with an apparatus $\mathcal{A}$, then it takes on a state of the form

$$\sum_n c_n |s_n\rangle |a_n\rangle \tag{6}$$

where $|s_n\rangle$ and $|a_n\rangle$ form bases of $\mathcal{S}$ and $\mathcal{A}$, respectively [3]. This correlation between the system and the apparatus seems at first glance to do what we want - it seems that performing a measurement using the apparatus and seeing what "pointer state" $|a_n\rangle$ it collapses into would tell us what state $|s_n\rangle$ our system collapses into as well. The problem of the preferred basis is that, in the common case, the state in Equation 6 holds for multiple bases of $\mathcal{S}$; in other words, the outcome of the measurement would hold for multiple, non-commuting observables, and *we do not know what basis of $\mathcal{S}$ we measured* *in*. This is fundamentally troubling, as it suggests we cannot learn what we want about our system.

One of the biggest reasons for decoherence's wide presence in recent discussions of quantum mechanics is because the preferred basis problem is naturally mitigated when decoherence is included in our model of a system.

As discussed in Section I above, decoherence includes the environment surrounding a system in the model of that system. We take these environmental effects into account by transforming Equation 6 into:

$$\sum_n c_n |s_n\rangle |a_n\rangle |e_n\rangle \tag{7}$$

where $|e_n\rangle$ are states of the environment [3], which were previously assumed to be orthogonal because of the large number of degrees of freedom.

The inclusion of $|e_n\rangle$ allows for the application of the Tridecompositional Uniqueness Theorem [3], which states that if a composite system can be written in Schmidt form across three spaces (like Equation 7), then that form is unique. However, justification must be given for why Equation 7 is a valid form for the system's state, as the environment can have an enormous number of different possible states; this justification comes from the "einselection" (environment-induced superselection) [2] the environment imposes on the possible bases.

Einselection of a basis can be thought of in multiple different ways. Zurek states that the preferred basis is one that "contains a reliable record of the state of the system" [2]; Schlosshauer explains that "it's the basis in which the system-apparatus correlations $|s_n\rangle|a_n\rangle$ are left undisturbed by the subsequent formation of correlations with the environment (the stability criterion)" [3]. In other words, if the system-apparatus composite state $\mathcal{SA}$ in a given basis can transform into the form of Equation 7 upon interaction with the environment, it implies that the $\mathcal{SA}$ state in that basis was robust to that environmental interaction. Any other possible (non-preferred) basis would not survive the new environmental correlations. Mathematically, there will be some Hamiltonian $H_{\mathcal{SA}\leftrightarrow\mathcal{E}}$ that describes how the environment will interact with our specific $\mathcal{SA}$ (we have to determine that using our knowledge of this specific system and the nature of the interactions the environment will have with it). We can then find projection operators (P_A) onto the eigenbasis of A that commute with that Hamiltonian ($[H_{\mathcal{SA}\leftrightarrow\mathcal{E}}, P_{\mathcal{A}}] = 0$) [2].

This einselection can also be discussed in more physical terms; Schlosshauer [3] provides an example. Consider a quantum system with various energy states, and the gap between the different energy states is greater than the maximum energy the environment could nominally provide to the system. In this case, einselection will select the energy eigenstates as the robust or stable basis; we can thus build an apparatus to measure this system along an energy-based observable. Different systems with different environmental interaction Hamiltonians could lead to the ability to perform measurements on different bases, observing other classical outcomes like position eigenstates instead.

Given this superselection, we've established Equation 7 as a valid form for the post-decoherence state of our open quantum system and thus have a unique basis of $\mathcal{S}$ along which to measure.

III. Consequences for QM Interpretations

A topic that we find quite interesting, and that Schlosshauer addresses heavily [3], is the potential for the decoherence theory to change the landscape of interpretations that exist for how the formalism of quantum mechanics applies to our universe. These interpretations largely focus on how the "measurement problem" is solved - how states of superpositions lead to classical, definite outcomes. There are a number of interpretations - standard/Copenhagen, modal, relative-state, pilot-wave, and a number of others - and decoherence could play a fundamental role in their feasibility, inter-operation, or potential demise.

Schlosshauer describes ways decoherence could potentially interact with the interpretations; here, we paraphrase them and provide examples of the interaction.

1. Decoherence could remove the need for certain interpretive additions to quantum mechanics. Often, the greatest detractors of quantum mechanical interpretations focus on the additions the interpretations need to make to the quantum mechanical formalism in order to "work". Decoherence can be seen as extending the reach of the formalism to bring us closer to explaining the classical outcomes we observe without deviating in any way from the formalism. Relative-state theories (explained more thoroughly in 4 below) have a similar goal and therefore can be inclined to adhere to the decoherence program; the success of decoherence could lead to the success of those types of more naturally-arising theories, directly reducing the scope of interpretive additions necessary.

2. Decoherence could protect an interpretation from empirical disproof. An example of this comes in physical collapse theories, which include the presumption of additional fundamental physical processes ("reductions") that lead to superpositions evolving into classical, determinate states over time. These theories predict evolution equations that are extremely similar to those established through the decoherence theory. Because decoherence has been thoroughly experimentally confirmed, this similarity in their evolution predictions allows for a protection of physical collapse theories from empirical disproof, at least temporarily.

3. Decoherence could disprove a currently-feasible interpretation. Opposing 1 and 2 above, decoherence as an extension of the quantum mechanical formalism could render interpretations impossible in reality. The Copenhagen interpretation is likely a victim of this. Copenhagen takes the "standard" theory that wavefunction collapse is an exception to an otherwise unitary-driven time evolution, and additionally supposes that "classicality is not to be derived from quantum mechanics" [3]. In addition to breaking from the majority of established physical theories, this hypothesis of some sharp disconnection between microscopic and macroscopic physics is much less feasible when decoherence effects are considered. Quoting Schlosshauer once more, "it is reasonable to anticipate that decoherence... could lead to a complete and consistent derivation of the classical world from quantum mechanical principles." The results from theoretical and empirical review of decoherence effects point towards the absence of a clear-cut quantum-classical boundary. If the theory garners more success, the Copenhagen interpretation will fade away more than it already has.

4. Decoherence could "physically motivate" currently abstract assumptions made by interpretations. This helps boost an interpretation's feasibility, as physically motivated ideas obviously outshine ad hoc assumptions. Relative-state theories, which invoke a "branching" or "splitting" mechanism to explain how one classical outcome is observed upon measuring a state in a superposition, benefit from this motivation. These theories include commonly referenced ones like the many-worlds interpretation (MWI). The branching or splitting in a theory like MWI explains definite outcomes, but requires the problem of the preferred basis to be solved in order to identify the basis along which that branching occurs. These theories therefore benefit from decoherence because einselection effectively solves that problem, allowing for the assumption of an environment-selected basis in which the branching can occur.

IV. Applications to Quantum Circuits

To connect the theory of quantum decoherence to quantum computation, we looked at the 2019 study from A. Ash-Saki et al. [4]. In this work, the authors investigate the impact of environment-induced decoherence on quantum circuits through simulations that model different quantum noise effects. They show that the quantum gate in question, the choice of quantum noise model, and the input state all affect the resilience of the system to decoherence effects.

Instead of using the operator-sum representation to model the evolution of their quantum systems, Ash-Saki et al. employ the so-called "master equation", given as:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_j [2L_j\rho L_j^\dagger - L_j^\dagger L_j \rho - \rho L_j^\dagger L_j] \tag{8}$$

where the L_j are the Linbald operators which describe the interaction between the system and the environment [4]. This form of the master equation is appropriately called the Linbald form, and describes how the density matrix ρ changes over time due to the Hamiltonian H and the Linbald operators L_j.

As the system evolves, environment-induced decoherence effects cause it to vary from its original state. To quantify this change, the authors use fidelity, which is defined by Nielsen and Chuang [5] as:

$$F(\rho,\sigma) \equiv \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \tag{9}$$

Using this metric, if two states ρ and σ are identical their fidelity $F(\rho,\sigma) = 1$. If instead the two states are orthogonal their fidelity $F(\rho,\sigma) = 0$. The fidelity shown on the vertical axis of the plots below compares the initial state of the system to the state of the system after decoherence effects have decayed the initial state. Ash-Saki et al. discuss two different quantum noise models in their study, amplitude damping and phase damping [4], and we explore these effects below.

A. Amplitude Damping

We use amplitude damping as a general model for the loss of energy of a quantum system to environmental noise. This effect is modeled by the following Linbald operator:

$$L_{amp} = \sqrt{\gamma}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{10}$$

where γ is a time constant of the system, the rate of spontaneous emission [4]. This operator acts on a single qubit, and we can see from its matrix form that on a state in the computational basis, L_{amp} decays $|1\rangle$ components to $|0\rangle$ components. This effect can be seen in Fig. 3 below for input state $|110\rangle$.

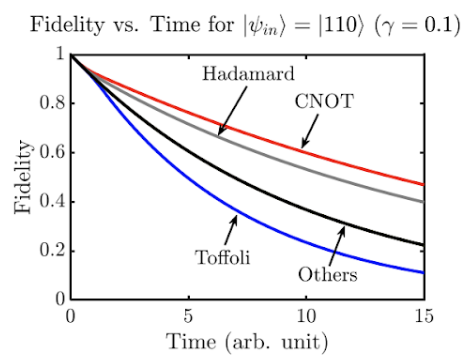

Fig. 3. Amplitude Damping with Input State $|110\rangle$. From Ash-Saki et al. [4]

Operations performed with each gate lose fidelity over time as energy dissipates to the environment and the portions of the state in the $|1\rangle$ direction rotate towards the $|0\rangle$ direction.

This trial also demonstrates the effect of the choice of gate on the system's fidelity over time. Take for example the CNOT gate given by the red curve in Fig. 3. This gate takes the first qubit of the state as the control qubit and flips the second qubit if the control qubit is $|1\rangle$. In this case, for the input state $|110\rangle$, the first (control) qubit is $|1\rangle$, so the CNOT gate flips the second qubit from $|1\rangle$ to $|0\rangle$, giving the state $|100\rangle$. Because amplitude damping only acts on $|1\rangle$ components and the CNOT gate removed a $|1\rangle$ from this state by making the transformation $|110\rangle \to |100\rangle$, amplitude damping has the least effect on this transformed state when compared to the other gates' outputs, since there are fewer $|1\rangle$ components in the final state. Fig. 3 reinforces this result as the CNOT gate has the smallest fidelity loss of all the gates studied.

In contrast, if the input state were $|100\rangle$, as in Fig. 4, the CNOT gate would make the transformation $|100\rangle \to |110\rangle$ and, due to the increase of $|1\rangle$ states, we expect greater fidelity loss.

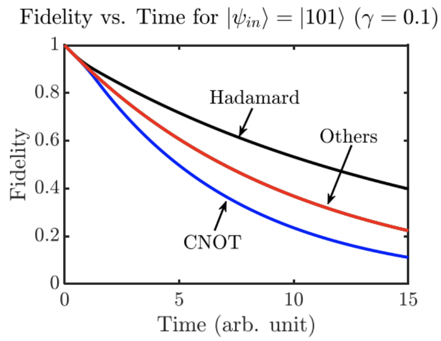

Fig. 4. Amplitude Damping with Input State $|100\rangle$. From Ash-Saki et al. [4]

This effect is shown clearly in Fig. 4, as the blue CNOT curve decays more quickly than any other gate, and much more quickly than it did in Fig. 3 where the input state was $|110\rangle$.

B. Phase Damping

To model quantum decoherence effects that do not draw energy from the system, we use phase damping instead of amplitude damping. The Linbald operator for this quantum noise model is given by:

$$L_{phase} = \sqrt{\lambda}\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} \tag{11}$$

where λ is a constant that defines the strength of the damping effect [4]. Phase damping has an effect similar to the theoretical model of decoherence we discussed in section II.A above, in that it eliminates the off-diagonal interference terms of the density operator. This makes the duration of the effect potentially finite, as once the off-diagonal terms decay to 0, phase damping no longer changes the state. This can be clearly seen in Fig. 5 below. For several gates, including the CNOT gate discussed previously, fidelity loss happens very quickly and does not decrease after a certain point. This corresponds to complete diagonalization of the density operator and the emergence of classical probabilities.

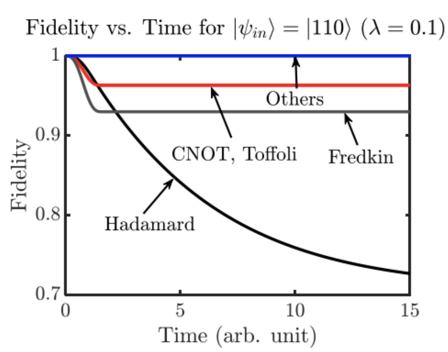

Fig. 5. Amplitude Damping with Input State $|110\rangle$. From Ash-Saki et al. [4]

Phase damping also demonstrates that the choice of gate has an impact on the change in fidelity of the system. For some gates that initially diagonalize the state, as shown by the blue line in Fig. 5, phase damping has no effect since there are already no off-diagonal interference terms. Other gates that flip single qubits between basis states, such as CNOT and Toffoli (control not with two control bits), show rapid fidelity loss until the state is diagonalized, after which no change occurs. Finally, the Hadamard gate, which creates a uniform superposition of basis states, will constantly create interference terms as the system evolves, and never become fully diagonalized. This is also clearly shown by Fig. 5, as the black line representing the Hadamard gate decays consistently throughout the time period, unlike the other gates studied.

V. Conclusions

Quantum decoherence presents a robust framework for analyzing quantum systems when environmental effects are considered. In the context of the quantum mechanical formalism, it provides solutions to long-standing problems; in the context of quantum computing, it provides an obstacle for researchers to overcome.

Acknowledgment

We would like to thank Professor Sandeep Pradhan for running an enlightening and interesting course on quantum information, probability, and computing.

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